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On the best interval quadrature formulae for classes of differentiable periodic functions

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Dedicated to Henryk Wozniakowski on the occasion of his 60th birthday

Abstract

In this paper we solve the problem about optimal interval quadrature formula for the class $W^r F$ of differentiable periodic functions with rearrangement invariant set F of their derivatives of order r. We prove that the formula with equal coefficients and n node intervals having equidistant midpoints is optimal for considering classes. To this end a sharp inequality for antiderivatives of rearrangements of averaged monosplines is proved.

Keywords: Quadrature formulae; Monosplines; Rearrangements

1. Introduction, notations, statement of the problem

Let L_p, $1 \leqslant p \leqslant \infty$, be the space of 2π-periodic functions $f : \mathbb{R} \to \mathbb{R}$ with the usual norm

$$\|f\|_p = \begin{cases} \left(\int_0^{2\pi} |f(t)|^p \, dt\right)^{1/p} & \text{if } p < \infty, \\ \operatorname{esssup}\{|f(t)| : t \in [0, 2\pi)\} & \text{if } p = \infty. \end{cases}$$

Let also $C_{2\pi}$ be the space of continuous 2π-periodic functions $f : \mathbb{R} \to \mathbb{R}$ endowed with the uniform norm $\|f\|_C$.

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Denote by K_n, $n = 1, 2, \dots,$ the set of all possible quadrature formulae of the form

$$\kappa(f) = \sum_{j=1}^{n} a_j f(x_j),$$

where $x_1 < x_2 < \cdots < x_n < x_1 + 2\pi$, $a_j \in \mathbb{R}$.

Let $\mathcal{M}$ be some (non-symmetric in general) class of continuous 2π-periodic functions. For $f \in \mathcal{M}$ and $\kappa \in K_n$ set

$$R(f, \kappa) = \int_0^{2\pi} f(t)\,dt - \kappa(f).$$

The error of approximate integration with the help of the formula $\kappa \in K_n$ on the class $\mathcal{M}$ we shall characterize by the pair of values

$$R^{\pm}(\mathcal{M}, \kappa) = \sup\{R(\pm f, \kappa) : f \in \mathcal{M}\}$$

or, equivalently, with the help of the interval

$$\Delta(\mathcal{M}, \kappa) := [-R^{-}(\mathcal{M}, \kappa), R^{+}(\mathcal{M}, \kappa)].$$

Certainly, for the symmetric classes $\mathcal{M}$ we have $R^{+}(M, \kappa) = R^{-}(M, \kappa)$.

Set

$$\mathcal{R}^{\pm}(\mathcal{M}, K_n) = \inf\{R^{\pm}(\mathcal{M}, \kappa) : \kappa \in K_n\}. \tag{1.1}$$

The Kolmogorov problem about the best quadrature formula for the class $\mathcal{M}$ can be formulated in the following way. Find the values (1.1) and find the formulae $\kappa \in K_n$ that realize the infimum in the right hand part of (1.1), if such formulae exist. The case when there exists a quadrature formula $\overline{\kappa}$, which realizes infimum in both $\mathcal{R}^{+}(\mathscr{M}, K_n)$ and $\mathcal{R}^{-}(\mathscr{M}, K_n)$, is especially interesting. For this $\overline{\kappa}$ and for an arbitrary formula κ we shall have

$$\Delta(\mathcal{M}, \overline{\kappa}) \subset \Delta(\mathcal{M}, \kappa).$$

Quadrature formula satisfying the latter conditions will be called optimal for the class $\mathcal{M}$.

Let $0 < h < \pi/n$ be given. Denote by $K_n^i(h)$ the set of so-called interval quadrature formulae of the form

$$\kappa^i(f) = \sum_{j=1}^{n} b_j \frac{1}{2h} \int_{y_j-h}^{y_j+h} f(t)\,dt,$$

where $y_1 < y_2 < \cdots < y_n < y_1 + 2\pi$, $b_j \in \mathbb{R}$.

For $f \in \mathcal{M}$ and $\kappa^i \in K_n^i(h)$ set

$$R(f, \kappa^i) = \int_0^{2\pi} f(t)\,dt - \kappa^i(f).$$

The error of approximate integration with the help of $\kappa^i \in K_n^i(h)$ on the class $\mathcal{M}$ we shall characterize by the pair of values

$$R^{\pm}(\mathscr{M}, \kappa^i) = \sup\{R(\pm f, \kappa^i) : f \in \mathscr{M}\}$$

or, that is equivalent, with the help of the interval

$$\Delta(\mathcal{M}, \kappa^i) := [-R^-(\mathcal{M}, \kappa^i), R^+(\mathcal{M}, \kappa^i)].$$

As above, for symmetric classes $\mathcal{M}$ we have $R^+(M, \kappa^i) = R^-(M, \kappa^i)$.

Set

$$\mathcal{R}^{\pm}(\mathcal{M}, K_n^i(h)) = \inf\{R^{\pm}(\mathcal{M}, \kappa^i) : \kappa^i \in K_n^i(h)\}. \tag{1.2}$$

The analog of the Kolmogorov problem about the best interval quadrature formula for the class $\mathcal{M}$ can be formulated in the following way. Find the values (1.2) and find the formulae $\kappa^i \in K_n^i(h)$ that realizes the infimum in the right hand part of (1.2). For the interval formulae as well as for usual quadrature formulae the case when there exists an interval quadrature formula $\overline{\kappa^i}$ which realize infimum in both $\mathcal{R}^+(\mathscr{M}, K_n^i(h))$ and $\mathcal{R}^-(\mathscr{M}, K_n^i(h))$ is especially interesting. For this $\overline{\kappa^i}$ and for an arbitrary formula κ^i we shall have

$$\Delta(\mathcal{M}, \overline{\kappa^i}) \subset \Delta(\mathcal{M}, \kappa^i).$$

Interval quadrature formula satisfying the latter conditions will be called optimal for the class $\mathcal{M}$.

From the applications point of view, interval quadrature formulae are more natural than the usual quadrature formulae based on values at points, since quite often the result of measuring physical quantities, due to the structure of the measurement devices, is an average values of the function, describing the studied quantities, over some interval. Note that one can obtain the usual quadrature formula from the corresponding interval quadrature formula as a limit case, setting $h \to 0$.

Given $h > 0$, define the Steklov operator $S_h : L_1 \to C_{2\pi}$ in the following way:

$$S_h(f)(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(t)\, dt.$$

We shall often write f^h instead of $S_h(f)$.

It can be easily seen that the problem of finding the optimal interval quadrature formula for the class $\mathcal{M}$ can be considered as a problem of finding the optimal usual quadrature formula for the class $S_h(\mathcal{M}) := \{S_h(f) : f \in \mathcal{M}\}$.

Let $f \in L_1$. The notation $f \perp 1$ means that

$$\int_0^{2\pi} f(t)\, dt = 0.$$

Let F be a subset of L_1 such that $\{f \in F : f \perp 1\} \neq \emptyset$. For $r = 1, 2, \ldots$ denote by $W^r F$ the class of functions f that have locally absolutely continuous derivative $f^{(r-1)}$ and such that $f^{(r)} \in F$. In the case when F is the unit ball of the space L_p we obtain the standard Sobolev class W_p^r of periodic functions.

For a non-negative function $f \in L_1$ let us denote by $P(f, t)$ the decreasing rearrangement (see e.g. [9, p. 130, 10, pp. 92, 93]) of the restriction of f to $[0, 2\pi)$. If g is an arbitrary function from L_1, then set (see e.g. [10, p. 99])

$$\Pi(g, t) = P(g_+, t) - P(g_-, 2\pi - t),$$

where $g_{\pm}(t) = \max\{\pm g(t); 0\}$. The set $F \subset L_1$ is called rearrangement invariant or, shortly, Π-invariant if conditions $f \in F$ and $\Pi(g) = \Pi(f)$ imply $g \in F$.

In order to illustrate the variety of the classes $W^r F$ with Π-invariant sets F we mention some examples.

1. For F one can take the unit sphere of any symmetric space of 2π-periodic functions embedded in L_1, in particular, the unit sphere in the space L_p, $1 \leqslant p \leqslant \infty$, in Orlich [11], Lorentz and Marcinkiewicz [12,22] spaces.
2. Let ϕ be an arbitrary non-negative, non-decreasing function defined on $[0, \infty)$. One can take
$$F = F(\phi) = \left\{ f \in L_1 : \int_0^{2\pi} \phi(|f(t)|)\, dt \leqslant 1 \right\}.$$
3. Let $\gamma, \delta > 0$ be non-negative real numbers, $1 \leqslant p \leqslant \infty$. One can take
$$F = F_{p;\gamma,\delta} = \{\|\gamma f_+ + \delta f_-\|_p \leqslant 1\}.$$
We shall denote the corresponding class $W^r F_{p;\gamma,\delta}$ by $W^r_{p;\gamma,\delta}$.
4. Very interesting classes $W^r F_{f,\Pi}$ correspond to the set $F_{f,\Pi} = \{g \in L_1 : \Pi(g) = \Pi(f)\}$, where f is a fixed function from L_1, $f \perp 1$.
5. For F one can take the set
$$F_{f,P} = \{g \in L_1 : P(|g|, t) = P(|f|, t),\ t \in [0, 2\pi)\}$$
or
$$F'_{f,P} = \{g \in L_1 : P(|g|, t) \leqslant P(|f|, t),\ t \in [0, 2\pi)\}.$$

The list of examples could, of course, be continued.

The following integral representation for functions $f \in W^r F$ plays an essential role in investigation of various extremal problems for classes $W^r F$. Let

$$D_r(x) = \frac{1}{\pi} \sum_{j=1}^{\infty} j^{-r} \cos(jx - \pi r/2), \quad r \in \mathbb{N}$$

be the Bernoulli kernel. Then

$$f(x) = \frac{a_0}{2} + \int_0^{2\pi} D_r(x-t) f^{(r)}(t)\, dt = \frac{a_0}{2} + (D_r * f^{(r)})(x), \tag{1.3}$$

where

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(t)\, dt.$$

Note that, considering the problem on optimization of quadrature formulae or interval quadrature formulae for the classes $W^r F$, we may restrict our consideration by formulae κ from K_n such that $\sum_{j=1}^n a_j = 2\pi$ or by formulae κ^i from $K_n^i(h)$ such that $\sum_{j=1}^n b_j = 2\pi$ only. For such formulae set

$$m(t) = m_{\kappa,r}(t) = -\sum_{j=1}^n a_j D_r(x_j - t)$$

and

$$m^h(t) = m^h_{\kappa^i,r}(t) = -\sum_{j=1}^{n} b_j D_r^h(y_j - t).$$

Set $M_n^r := \{m_{\kappa,r} : \kappa \in K_n\}$, and $M_n^{r,h} = S_h(M_n^r) := \{m^h_{\kappa^i,r} : \kappa^i \in K_n^i(h)\}$. Functions from M_n^r and from $M_n^{r,h}$ will be called monosplines and averaged monosplines, respectively.

With the help of representation (1.3) one can obtain the error of approximate integration by these formulae in the form

$$R(f,\kappa) = \int_0^{2\pi} f^{(r)}(t)m(t)\,dt, \quad m = m_{\kappa,r} \in M_n^r$$

if $\kappa \in K_n$, or in the form

$$R(f,\kappa^i) = \int_0^{2\pi} f^{(r)}(t)m^h(t)\,dt, \quad m^h = m^h_{\kappa^i,r} \in M_n^{r,h} \tag{1.4}$$

if $\kappa^i \in K_n^i(h)$.

Denote by $S_n^r(\gamma,\delta)$, $n = 1,2,\ldots$, $r = 0,1,\ldots$, $\gamma,\delta > 0$, the set of functions $f \in W^r_{\infty;\gamma,\delta}$ with zero mean value on a period such that $\gamma^{-1}(f^{(r)})_+ + \delta^{-1}(f^{(r)})_- \equiv 1$ and $f^{(r)}$ admits at most $2n$ changes of sign on a period.

In this paper we shall discuss the Kolmogorov problems on optimal quadrature formulae and optimal interval quadrature formulae for classes W^rF with Π-invariant sets F. We shall show that for any fixed $h \in (0,\pi/n)$ the interval quadrature formula having equidistant nodes y_j, $j = \overline{1,n}$, and equal coefficients $b_j = 2\pi/n$ is optimal for the class W^rF among all interval quadrature formulae from $K_n^i(h)$. To this end a sharp inequality for antiderivatives of rearrangements of averaged monosplines will be proved.

The paper is organized in the following way. In Section 2 we shall present the known results, formulate main results of the paper, and describe the ideas of the proof. Some auxiliary results will be presented in Section 3. In Sections 4–7 we shall prove results, formulated in Section 2.

2. Background, main results, scheme of the proof

Set

$$\kappa_n(f) = \frac{2\pi}{n}\sum_{j=1}^{n} f(2\pi j/n)$$

and

$$\kappa_n^i(f) = \frac{2\pi}{n}\sum_{j=1}^{n}\frac{1}{2h}\int_{2\pi j/n-h}^{2\pi j/n+h} f(t)\,dt.$$

In addition, set

$$m_{n,r}(x) = -\frac{2\pi}{n}\sum_{j=1}^{n} D_r\left(\frac{2\pi j}{n} - x\right)$$

and denote by $\phi_{n,r;\gamma,\delta}$ the rth periodic integral with zero mean value over a period of a $2\pi n^{-1}$-periodic function $\phi_{n,0;\gamma,\delta}$ which equals γ on the interval $[0, 2\pi\delta n^{-1}(\gamma+\delta)^{-1})$, and equals $-\delta$ on the interval $[2\pi\delta n^{-1}(\gamma+\delta)^{-1}, 2\pi n^{-1})$.

It was proved in the papers of Motornyi [16], Ligun [14], and Zhensykbaev (see [23,24]) that if $\mathcal{M} = W_p^r,\ r = 1, 2, \ldots,\ 1 \leqslant p \leqslant \infty$, then

$$\mathcal{R}^{\pm}(\mathcal{M}, K_n) = R^{\pm}(\mathcal{M}, \kappa_n).$$

At the same time it does not hold for some natural analogues of the class W_p^r [19]. Therefore, it was an interesting problem to determine the most general conditions on the class $\mathcal{M}$ of functions that ensure the optimality of formula κ_n. This problem was solved by Babenko [3,5]. He proved the following:

Theorem A. *Let $n, r = 1, 2, \ldots$, and let $F \subset L_1$ be rearrangement invariant. Then*

$$\begin{aligned}\mathcal{R}^{\pm}(W^r F, K_n) &= R^{\pm}(W^r F, \kappa_n)\\ &= \sup\left\{\int_0^{2\pi} \Pi(\pm f,\ t)\Pi(m_{n,r}, t)\,dt\ :\ f \in F,\ f \perp 1\right\}.\end{aligned}$$

Let us describe the scheme of the proof of Theorem A.

For non-negative 2π-periodic functions f and F we shall write $f \prec F$ if for any $x \in [0, 2\pi]$,

$$\int_0^x P(f, t)\,dt \leqslant \int_0^x P(F, t)\,dt.$$

The following extremal property of monosplines was proved in [5] in order to establish Theorem A.

Theorem B. *Let $n, r = 1, 2, \ldots$. Then for any $m \in M_n^r$ and any $\lambda \in \mathbb{R}$,*

$$(m_{n,r} - \lambda)_{\pm} \prec (m - \lambda)_{\pm}.$$

To prove Theorem B it was enough (see Theorem 10 in Section 3) to prove the following:

Theorem C. *Let $n, r = 1, 2, \ldots$. Then for any $m \in M_n^r$ and any $\gamma, \delta > 0$,*

$$E_0(m_{n,r})_{1;\gamma,\delta} \leqslant E_0(m)_{1;\gamma,\delta}.$$

(For the definition of the values $E_0(f)_{1;\gamma,\delta}$, $f \in L_1$, see Section 3.)

To prove this it was enough to prove:

Theorem D. *Let $r = 1, 2, \ldots$ and $\gamma, \delta > 0$. Then for an arbitrary $n \in \mathbb{N}$ the quadrature formula with equidistant nodes and equal coefficients is optimal for the class $W^r_{\infty;\gamma,\delta}$. Moreover,*

$$\begin{aligned}\mathcal{R}^{\pm}(W^r_{\infty;\gamma,\delta}, K_n) = R^{\pm}(W^r_{\infty;\gamma,\delta}, \kappa_n) &= E_0(m_{n,r})_{1;\gamma^{-1},\delta^{-1}}\\ &= -2\pi \min_u(\pm\phi_{n,r;\gamma^{-1},\delta^{-1}}(u)).\end{aligned}$$

To prove the last theorem the following two theorems were established:

Theorem E. *For any n points* $x_1 < x_2 < \cdots < x_n < x_1 + 2\pi$ *there exists a spline* $g \in S_n^r(\gamma, \delta)$ *with equal minima at these points.*

Theorem F. *Let* $n, r = 1, 2, \ldots$ *and* $\alpha, \beta, \gamma, \delta > 0$. *Then for any* $g \in S_n^r(\gamma, \delta)$,

$$E_0(\phi_{n,r;\gamma,\delta})_{1;\alpha,\beta} \leqslant E_0(g)_{1;\alpha,\beta}.$$

Interval quadrature formulae have been considered by many mathematicians (see for instance [18,20,13,21,4,17,15]). The results about optimal interval quadrature formula for the classes of differentiable periodic functions are known for the classes W_1^r [4], W_∞^r [17], and $W^1 F$ [7,8].

The main result of our paper is the following:

Theorem 1. *Let* $n, r = 1, 2, \ldots$ *and* $0 < h < \pi/n$. *Then for an arbitrary* Π*-invariant set F*,

$$\begin{aligned} \mathcal{R}^{\pm}(W^r F, K_n^i(h)) &= R^{\pm}(W^r F, \kappa_n^i) \\ &= \mathcal{R}^{\pm}(S_h(W^r F), K_n) = R^{\pm}(S_h(W^r F), \kappa_n) \\ &= \sup\left\{\int_0^{2\pi} \Pi(\pm f,\ t)\Pi(S_h(m_{n,r}), t)\,dt \ :\ f \in F,\ f \perp 1\right\}. \end{aligned}$$

To prove this theorem we shall use the above presented scheme of the proof of Theorem A. In particular, we shall prove the following theorem which is of independent interest.

Theorem 2. *Let* $n, r = 1, 2, \ldots$ *and* $0 < h < \pi/n$. *Then for any* $m^h \in M_n^{r,h}$ *and any* $\lambda \in \mathbb{R}$,

$$(S_h(m_{n,r}) - \lambda)_{\pm} \prec (m^h - \lambda)_{\pm}.$$

To prove Theorem 2 it is enough to prove:

Theorem 3. *Let* $n, r = 1, 2, \ldots$. *Then for any* $m^h \in M_n^{r,h}$ *and any* $\gamma, \delta > 0$,

$$E_0(S_h(m_{n,r}))_{1;\gamma,\delta} \leqslant E_0(m^h)_{1;\gamma,\delta}.$$

To prove this it suffices to prove the following:

Theorem 4. *Let* $r = 1, 2, \ldots$ *and* $\gamma, \delta > 0$. *Then for an arbitrary* $n \in \mathbb{N}$ *the interval quadrature formula with equal coefficients and node intervals having equidistant midpoints is optimal for the class* $W^r_{\infty;\gamma,\delta}$. *Furthermore,*

$$\begin{aligned} \mathcal{R}^{\pm}(W^r_{\infty;\gamma,\delta}, K_n^i(h)) = R^{\pm}(W^r_{\infty;\gamma,\delta}, \kappa_n^i) &= E_0(S_h(m_{n,r}))_{1;\gamma^{-1},\delta^{-1}} \\ &= -2\pi \min_u(\pm\phi^h_{n,r;\gamma^{-1},\delta^{-1}}(u)). \end{aligned}$$

To prove Theorem 4 we shall prove the following two theorems.

Theorem 5. *For every system of points* $x_1 < x_2 < \cdots < x_n < 2\pi + x_1$ *there exists a function* $f_r \in S_n^r(\gamma, \delta)$ *such that* f_r^h *attains equal minimal values at these points.*

Theorem 6. *Let $n, r = 1, 2, \dots, 0 < h < \pi/n$ and $\alpha, \beta, \gamma, \delta > 0$. Then for every $f \in S_n^r(\gamma, \delta)$,*

$$E_0(\phi^h_{n,r;\gamma,\delta})_{1;\alpha,\beta} \leqslant E_0(S_h(f))_{1;\alpha,\beta}.$$

The implementation of this outline meets serious difficulties connected with the fact that Steklov operator S_h does not have the following property: for any 2π-periodic function f having zero mean value on a period

$$\nu(S_h(f)) \leqslant \nu(f),$$

where $\nu(f)$ is number of sign changes of the function f on a period.

To overcome these difficulties we shall prove the following theorem which plays the crucial role in proofs of Theorems 5 and 6.

Theorem 7. *Let $n, r = 1, 2, \dots$ and $\gamma, \delta > 0$. Let splines $s_1, s_2 \in S_n^0(\gamma, \delta)$ be such that $\nu(s_1^h) = \nu(s_2^h) = 2n$. If $f(t) = s_1(t) - s_2(t)$ then $\nu(f^h) \leqslant \nu(f)$.*

3. Some auxiliary results

Here we shall present some known definitions and results which will be frequently used in the rest of the paper.

Let $1 \leqslant p \leqslant \infty$. Let $f \in L_p$ and let H be a subspace of L_1. We shall denote by $E(f; H)_p$ the best approximation of the function f by the subspace H in the L_p-metric, i.e.:

$$E(f; H)_p = \inf\{\|f - u\|_p \ : \ u \in H\}.$$

In addition, let

$$E^{\pm}(f; H)_p = \inf\{\|f - u\|_p \ : \ \pm u \leqslant \pm f, \ u \in H\}$$

denote the best one-sided approximation of the function f by the subspace H in the L_p-metric.

Let $\alpha, \beta > 0$. Then we shall denote by $E(f; H)_{p;\alpha,\beta}$ the best (α, β)-approximation [2] of the function f by the subspace H in the L_p-metric, i.e.:

$$E(f; H)_{p;\alpha,\beta} = \inf\{\|\alpha(f - u)_+ + \beta(f - u)_-\|_p : u \in H\}.$$

For $\alpha = \beta$ we obtain, up to a constant factor, the usual best approximation (instead of $E(f; H)_{p;1,1}$ we shall write $E(f; H)_p$). By virtue of Theorem 2 in [2], as $\beta \to \infty$ ($\alpha \to \infty$), $E(f; H)_{p;1,\beta}$ ($E(f; H)_{p;\alpha,1}$) tends monotone non-decreasingly to the best approximation from below (from above) of the function f by the elements of H in the L_p-metric: $E^+(f; H)_p$ ($E^-(f; H)_p$), i.e.:

$$\lim_{\beta\to\infty} E(f; H)_{p;1,\beta} = E^+(f; H)_p \ \left(\lim_{\alpha\to\infty} E(f; H)_{p;\alpha,1} = E^-(f; H)_p\right).$$

This allows us to include the problem of the best approximation without constraint and the problem of the best one-sided approximation into the family of problems of the same type with "loose" constraints, and consider them from a general point of view (see for this reason also [3]). In what follows we shall allow $+\infty$ for α or β identifying $E(f; H)_{p;\alpha,\beta}$ with the corresponding one-sided approximation.

When H is the space of all constants, let $E_0(f)_{p;\alpha,\beta} = E(f; H)_{p;\alpha,\beta}$.

Theorem 8 (*Criterion for the best* (α, β)*-approximation,[2], Theorem 4*). *Let H be a finite dimensional subspace of L_p, $1 \leqslant p < \infty$, and $\alpha, \beta > 0$. For an element $u_0 \in H$ to be the best (α, β)-approximation for $f \in L_p$ in the L_p-metric, it is sufficient and (for $p = 1$ in the case if $f - u_0$ almost everywhere differs from 0) necessary, that for any $u \in H$,*

$$\int_0^{2\pi} u(t)|f(t) - u_0(t)|^{p-1}[\alpha^p \operatorname{sign}(f(t) - u_0(t))_+ - \beta^p \operatorname{sign}(f(t) - u_0(t))_-]\, dt = 0.$$

Theorem 9 (*Duality theorem for the best* (α, β)*-approximation, [2], Theorem 5*). *Let $1 \leqslant p < \infty$ and let H be any finite dimensional subspace of L_p. Then for any function $f \in L_p$,*

$$E(f; H)_{p;\alpha,\beta} = \sup\left\{\int_0^{2\pi} f(t)g(t)\, dt : \|\alpha^{-1}g_+ + \beta^{-1}g_-\|_q \leqslant 1,\ g \perp H\right\},$$

where $p^{-1} + q^{-1} = 1$.

Let $f, g \in L_1$. The convolution of functions f and g is defined as

$$(f * g)(t) = \int_0^{2\pi} f(t - \tau)g(\tau)\, d\tau, \quad t \in [0, 2\pi).$$

Let $\varepsilon > 0$ and $x \in [0, 2\pi)$. Define

$$A_\varepsilon(x) = \frac{1}{2\pi} \sum_{q=-\infty}^{\infty} \frac{e^{iqx}}{\operatorname{ch}(q\varepsilon)}.$$

It is easy to verify that the convolution of function $A_\varepsilon(x)$ and arbitrary periodic function is analytic on a real line. Hence, a convolution of $A_\varepsilon(x)$ and an arbitrary function, not identically constant, differs from zero almost everywhere. It is known (see, for example, [6]) that for every function $f \in C_{2\pi}$,

$$\nu(A_\varepsilon * f) \leqslant \nu(f). \tag{3.1}$$

In addition, for every $f \in C_{2\pi}$,

$$\|(A_\varepsilon * f)(\cdot) - f(\cdot)\|_{C_{2\pi}} \to 0$$

as $\varepsilon \to 0$.

Let $n, r = 1, 2, \ldots$ and $0 < h < \pi/n$. Due to Lemma 5.1 from [5], it is easy to verify that the following lemma holds.

Lemma 3.1. *Let the spline $g \in S_n^r(\gamma, \delta)$ with nodes at the points $x_1, \ldots, x_{2l}$ be such that $g^{(r)}(x) = \gamma$ for $x \in (x_1, x_2)$. Then*

$$\begin{aligned}(A_\varepsilon * g^h)(x) &= ((A_\varepsilon * D_r^h) * g^{(r)})(x)\\ &= (\gamma + \delta)\sum_{j=1}^{2l}(-1)^j \int_0^{2\pi} D_1(x_j - t)[D_r * A_\varepsilon]^h(x - t)\, dt.\end{aligned}$$

Lemma 3.2 (*Babenko [5], Lemma 5.2*). *Let the function $g \in L_1$ be almost everywhere different from every fixed constant and $g \perp 1$. Then*

$$E_0(g)_{1;\alpha,\beta} = \inf_{\lambda\in\mathbb{R}} \left[\frac{\alpha+\beta}{2} \int_0^{2\pi} |g(t)-\lambda|\,dt + 2\pi\lambda\frac{\beta-\alpha}{2} \right].$$

Lemma 3.3. *Let $s \in S_n^0(\gamma,\delta)$, and $x_1 < x_2 < \cdots < x_{2l} < x_1 + 2\pi$ be the nodes of s. Let $\lambda \in \mathbb{R}$, and*

$$F(x_1,\ldots,x_{2l};\lambda) = \int_0^{2\pi} \left| (\gamma+\delta)\sum_{j=1}^{2l}(-1)^j \int_0^{2\pi} D_1(x_j-t)[D_r * A_\epsilon]^h(x-t)\,dt - \lambda \right| dx.$$

Then F is continuously differentiable in the sense that the partial derivatives $\frac{\partial F}{\partial\lambda}$ and $\frac{\partial F}{\partial x_k}$, $k=\overline{1,2l}$, exist and are continuous. Moreover,

$$\frac{\partial F}{\partial \lambda} = -\int_0^{2\pi} \operatorname{sign}\left((\gamma+\delta)\sum_{j=1}^{2l}(-1)^j \int_0^{2\pi} D_1(x_j-t)[D_r * A_\epsilon]^h(x-t)\,dt - \lambda \right) dx,$$

$$\frac{\partial F}{\partial x_k} = (-1)^k(\gamma+\delta)\int_0^{2\pi} [A_\varepsilon * D_r]^h(x_k - x)$$
$$\times \operatorname{sign}\left((\gamma+\delta)\sum_{j=1}^{2l}(-1)^j \int_0^{2\pi} D_1(x_j-t)[D_r * A_\epsilon]^h(x-t)\,dt - \lambda \right) dx.$$

This lemma can be proved analogously to the proof of Lemma 5.3 from [5].

Lemma 3.4 (*See Babenko [6]*). *Let $n, r = 1, 2, \ldots,$ $\gamma, \delta > 0$, and $l \in \mathbb{N}$, $l < n$. Then for an arbitrary $t \in [0, 2\pi)$,*

$$\min_u (A_\varepsilon * \phi_{l,r;\gamma,\delta})(u) < (A_\varepsilon * \phi_{n,r;\gamma,\delta}(t)) < \max_u (A_\varepsilon * \phi_{l,r;\gamma,\delta})(u).$$

The statement of this lemma was noted in [6].

The following theorems represent the statements of Theorem 2.3 *and Lemmas* 2.2–2.3 *from* [5].

Theorem 10. *Let f and F be continuous 2π-periodic functions with zero mean value on a period and for all $\alpha, \beta > 0$ let*

$$E_0(f)_{1;\alpha,\beta} \leqslant E(F)_{1;\alpha,\beta}.$$

Then

$$f_\pm \prec F_\pm. \tag{3.2}$$

Theorem 11. *For any $f \in L_1$ with zero mean value on a period and for any $F \in L_1$ the following equality holds*:

$$\sup\left\{ \int_0^{2\pi} g(t)F(t)\,dt : \Pi(g) = \Pi(f) \right\} = \int_0^{2\pi} \Pi(f,t)\Pi(F,t)\,dt.$$

Theorem 12. *Let the 2π-periodic functions f and F be continuous with zero mean values on a period and such that for all $\lambda \in \mathbb{R}$ and $x \in [0, 2\pi)$, the inequality* (3.2) *holds. Then for any function $g \in L_1$ with zero mean value on a period we have*

$$\int_0^{2\pi} \Pi(g,t)\Pi(f,t)\,dt \leqslant \int_0^{2\pi} \Pi(g,t)\Pi(F,t)\,dt.$$

4. Some properties of averaged (γ, δ)-splines

In this section we shall prove Theorem 7 which plays very important role in the rest of the paper.

Let n be a positive integer, $\gamma, \delta > 0$, and $0 < h < \pi/n$. The following two results represent a generalization of Lemmas 2 and 3 from the paper of Motornyi [17] for the case of non-symmetric perfect splines.

Lemma 4.1. *Let $s \in S_n^0(\gamma, \delta)$ be an arbitrary spline and let us denote by $x_1 < x_2 < \cdots < x_{2n}$ its nodes on a period. Then the Steklov function s^h is non-decreasing on the interval $(x_j - h, x_{j+1} - h)$, if $s(t) \equiv \gamma$ on the interval (x_j, x_{j+1}), and is non-increasing on $(x_j - h, x_{j+1} - h)$, if $s(t) \equiv -\delta$ on (x_j, x_{j+1}).*

Proof. Let us consider the first derivative of the Steklov function

$$(s^h)'(t) = \frac{d}{dt}\left(\frac{1}{2h}\int_{t-h}^{t+h} s(t)\,dt\right) = \frac{1}{2h}[s(t+h) - s(t-h)].$$

This provides $(s^h)'(t-h) = [s(t) - s(t-2h)]/2h$. It can be easily seen that $(s^h)'(t-h) \geqslant 0$ on the interval $t \in (x_j, x_{j+1})$, if $s(t) \equiv \gamma$ on the same interval, and $(s^h)'(t-h) \leqslant 0$ on the interval $t \in (x_j, x_{j+1})$, if $s(t) \equiv -\delta$ on the same interval. Thus, we obtain s^h is non-decreasing on $(x_j - h, x_{j+1} - h)$, if $s(t) \equiv \gamma$ on the interval (x_j, x_{j+1}). Similarly, s^h is non-increasing, if $s(t) \equiv -\delta$ on the interval (x_j, x_{j+1}). This is the desired conclusion. □

Lemma 4.2. *Let $s \in S_n^0(\gamma, \delta)$. Assume that $\nu(s^h) = 2n$. Then the length of the interval (x_j, x_{j+1}) is greater than $2h\delta/(\gamma + \delta)$ in the case $s(t) \equiv \gamma$ on this interval, and is greater than $2h\gamma/(\gamma + \delta)$ in the case $s(t) \equiv -\delta$ on (x_j, x_{j+1}).*

Proof. Let $x_1 < x_2 < \cdots < x_{2n} < x_1 + 2\pi$ denote the nodes of the spline s and let $x_{2n+1} = x_1 + 2\pi$. Since $\nu(s^h) = 2n$, by the previous lemma, we have that $s^h(x_j - h)s^h(x_{j+1} - h) < 0$, $j = \overline{1, 2n-1}$. Without loss of generality, we may assume $\operatorname{sign} s^h(x_j - h) = (-1)^j$, $j = \overline{1, 2n}$. From this it follows that $s(t) \equiv \gamma$ on the interval (x_1, x_2).

Note that the sum of lengths of all intervals (x_j, x_{j+1}), $j = \overline{1, 2n}$, on which s attains the value γ, is equal to $2\pi\delta/(\gamma + \delta)$. Then there exists an interval on which $s(t) \equiv \gamma$, with the length greater than $2h\delta/(\gamma + \delta)$. Similarly, there exists an interval on which $s(t) \equiv -\delta$, with the length greater than $2h\gamma/(\gamma + \delta)$.

Suppose the assertion of the lemma is false. Then, due to the remark above, we obtain two possible cases:

(1) There exists $1 \leqslant j \leqslant 2n$ such that $s(t) \equiv \gamma$ for $t \in (x_{j-1}, x_j)$, $s(t) \equiv -\delta$ for $t \in (x_j, x_{j+1})$ and the length of the interval (x_{j-1}, x_j) is greater than $2h\delta/(\gamma + \delta)$ and the length of the interval (x_j, x_{j+1}) is less than $2h\gamma/(\gamma + \delta)$.

(2) There exists $1 \leqslant j \leqslant 2n$ such that $s(t) \equiv -\delta$ for $t \in (x_{j-1}, x_j)$, $s(t) \equiv \gamma$ for $t \in (x_j, x_{j+1})$ and the length of the interval (x_{j-1}, x_j) is greater than $2h\gamma/(\gamma+\delta)$ and the length of the interval (x_j, x_{j+1}) is less than $2h\delta/(\gamma+\delta)$.

We consider the first case in detail. The second one can be studied similarly. Without loss of generality, we may assume that $j = 2$. From this we have $s^h(x_3 - h) < 0$, since $\operatorname{sign} s^h(x_3 - h) = -1$.

Let us consider $x_3 - 2h \leqslant x_2 - 2h\delta/(\gamma+\delta)$. Then

$$
\begin{aligned}
s^h(x_3 - h) &= \frac{1}{2h}\int_{x_3-2h}^{x_3} s(t)\,dt \\
&= \frac{1}{2h}\left(\int_{x_3-2h}^{x_2-2h\delta/(\gamma+\delta)} s(t)\,dt + \int_{x_2-2h\delta/(\gamma+\delta)}^{x_2} s(t)\,dt + \int_{x_2}^{x_3} s(t)\,dt\right) \\
&\geqslant \frac{1}{2h}\left[(-\delta)\cdot\left(x_2 - \frac{2h\delta}{\gamma+\delta} - x_3 + 2h\right) + \gamma\cdot\frac{2h\delta}{\gamma+\delta} - \delta\cdot(x_3 - x_2)\right] = 0.
\end{aligned}
$$

In the case $x_3 - 2h \geqslant x_2 - 2h\delta/(\gamma+\delta)$ we obtain

$$
s^h(x_3 - h) = \frac{1}{2h}\left(\int_{x_3-2h}^{x_2} s(t)\,dt + \int_{x_2}^{x_3} s(t)\,dt\right) = \frac{1}{2h}\cdot[2h\gamma - (\gamma+\delta)(x_3 - x_2)] > 0.
$$

Thus, $s^h(x_3 - h) \geqslant 0$, which contradicts the fact that $s^h(x_3 - h) < 0$. $\square$

The following statement is a trivial corollary of Lemma 4.2.

Lemma 4.3. *Let the spline $s \in S_n^0(\gamma, \delta)$ be such that $\nu(s^h) = 2n$. Then for an arbitrary point $x \in [0, 2\pi)$ spline s has at most two sign changes on the interval $(x - h, x + h)$.*

Due to Lemma 4.3, considering different possibilities for location of points, where splines s_1 and s_2 change their signs, we obtain that the following lemma holds.

Lemma 4.4. *Let splines $s_1, s_2 \in S_n^0(\gamma, \delta)$ be such that $\nu(s_1^h) = \nu(s_2^h) = 2n$ and let x be an arbitrary point from the interval $[0, 2\pi)$. Then the difference $f(t) = s_1(t) - s_2(t)$ has at most two sign changes on the interval $(x - h, x + h)$.*

Lemma 4.5. *Let splines $s_1, s_2 \in S_n^0(\gamma, \delta)$ be such that $\nu(s_1^h) = \nu(s_2^h) = 2n$. Assume there exists a point $x \in [0, 2\pi)$ such that the function $f(t) = s_1(t) - s_2(t)$ has exactly two sign changes on the interval $(x - h, x + h)$. Then there exists $\tilde{x} > 0$ such that the function f has exactly one sign change on the interval $(x + \tilde{x} - h, x + \tilde{x} + h)$. Moreover, $f^h(y) = f^h(x)$ for arbitrary $y \in [x, x + \tilde{x}]$.*

Proof. Let $x \in [0, 2\pi)$ satisfy conditions of the lemma. Analyzing different possibilities for location of nodes of splines s_1 and s_2 on the interval $[x - h, x + h]$, we conclude that the function f has exactly two sign changes on this interval only when both splines s_1 and s_2 have exactly two sign changes on the interval $[x - h, x + h]$ and there exists a neighborhood $U(x - h)$ of the point $x - h$ such that $s_1(t) \equiv \text{const}$ and $s_2(t) \equiv \text{const}$ when $t \in U(x - h)$, and $s_1(x - h)\cdot s_2(x - h) < 0$. Without loss of generality, we may assume that $s_1(x - h) = \gamma$ and $s_2(x - h) = -\delta$.

Let $x_{1,1}, x_{1,2}, x_{1,3}$ and $x_{1,4}$ be the neighboring nodes of the spline s_1 such that

$$x_{1,1} \leqslant x - h < x_{1,2} < x_{1,3} < x + h \leqslant x_{1,4}$$

and let $x_{2,1}, x_{2,2}, x_{2,3}$ and $x_{2,4}$ be the neighboring nodes of the spline s_2 such that

$$x_{2,1} \leqslant x - h < x_{2,2} < x_{2,3} < x + h \leqslant x_{2,4}.$$

Then $s_1(t) \equiv \gamma$ in the case $t \in (x_{1,1}, x_{1,2})$ or $t \in (x_{1,3}, x_{1,4})$ and $s_1(t) \equiv -\delta$ in the case $t \in (x_{1,2}, x_{1,3})$. At the same time $s_1(t) \equiv -\delta$ in the case $t \in (x_{2,1}, x_{2,2})$ or $t \in (x_{2,3}, x_{2,4})$ and $s_1(t) \equiv \gamma$ in the case $t \in (x_{2,2}, x_{2,3})$.

Set $\tilde{x} = \min\{x_{1,2} - x + h; x_{2,2} - x + h\}$. Without loss of generality we assume $\tilde{x} = x_{1,2} - x + h$. This implies the splines s_1 and s_2 are equal to γ and $-\delta$, respectively, on the interval $(x - h, x_{1,2})$. Thus we obtain $f(t) = \gamma + \delta$ for $t \in (x - h, x_{1,2})$.

At the same time splines s_1 and s_2 are equal to γ and $-\delta$, respectively, on the interval $(x + h, x + \tilde{x} + h)$. Indeed, applying Lemma 4.2 we obtain $x_{1,3} - x_{1,2} > 2h\gamma/(\gamma + \delta)$ and $x_{1,4} - x_{1,3} > 2h\delta/(\gamma + \delta)$. From the last inequalities we conclude that $x_{1,4} - x_{1,2} > 2h = x + h + \tilde{x} - x_{1,2}$, hence that $x + \tilde{x} + h < x_{1,4}$. Similarly, $x + \tilde{x} + h < x_{2,4}$.

By these arguments for an arbitrary $y \in [0, \tilde{x}]$ we have

$$f(x - h + y) = f(x - h) = \gamma + \delta = f(x + h) = f(x + h + y).$$

For every $z \in [0, \tilde{x}]$ it can be easily seen that $z < 2h$. It follows that the following equalities hold:

$$\begin{aligned} & f^h(x + z) - f^h(x) \\ &= \frac{1}{2h}\left(\int_{x-h}^{x+h} f(t)\,dt - \int_{x+z-h}^{x+z+h} f(t)\,dt\right) \\ &= \frac{1}{2h}\left(\int_{x-h}^{x+z-h} f(t)\,dt + \int_{x+z-h}^{x+h} f(t)\,dt - \int_{x+z-h}^{x+h} f(t)\,dt - \int_{x+h}^{x+z+h} f(t)\,dt\right) \\ &= \frac{1}{2h}\left(\int_0^z f(x - h + \tau)\,d\tau - \int_0^z f(x + h + \tau)\,d\tau\right) = 0. \end{aligned}$$

Obviously, the function f does not have more than two sign changes on the interval $[x + \tilde{x} - h, x + \tilde{x} + h]$, which completes the proof. □

Proof of Theorem 7. Set $\nu(f^h) = 2b$, where b is a positive integer. Due to Lemmas 4.4 and 4.5, there exist points $x_1 < x_2 < \cdots < x_{2b} < x_1 + 2\pi$ such that sign $f^h(x_j) = (-1)^j$, $j = \overline{1, 2b}$, and the function f has at most one sign change on each of intervals $[x_j - h, x_j + h]$, $j = \overline{1, 2b}$. Clearly, for every $j = \overline{1, 2b}$ there exists a non-empty interval $\Delta_j \subset [x_j - h, x_j + h]$ such that sign $f(t) = (-1)^j$ on it. Let us denote by y_j, y_j^* and y_j^{**} the midpoint, the left and right endpoints of the interval Δ_j, respectively. This implies $x_j - h < y_j < x_j + h$ for every $j = \overline{1, 2b}$.

We shall show that the sequence $\{y_j\}_{j=1}^{2b}$ increases and sign $f(y_j) = (-1)^j$, $j = \overline{1, 2b}$.

The second proposition holds by choosing points y_j. Suppose, there exists j_0 such that $y_{j_0} > y_{j_0+1}$. Without loss of generality we may take $j_0 = 1$. It can be easily seen that $y_2 \in (x_2 - h, x_2 + h)$ and we conclude from the assumption and inequality $x_1 - h < x_2 - h$ that $y_1 \in (x_2 - h, x_2 + h)$

and $y_2 \in (x_1 - h, x_1 + h)$. It is easy to verify that

$$x_1 - h < x_2 - h \leqslant y_2^* < y_2^{**} \leqslant y_1^* < y_1^{**} \leqslant x_1 + h < x_2 + h.$$

Thus, $f(t) \geqslant 0$ when $t \in (x_1 - h, x_2 - h)$, otherwise there exist three points from the interval $[x_1, -h, x_1 + h]$ with alternate sign. Similarly, $f(t) \leqslant 0$ when $t \in (x_1 + h, x_2 + h)$. Therefore,

$$0 < f^h(x_2) - f^h(x_1) = -\int_{x_1-h}^{x_2-h} f(t)\,dt + \int_{x_1+h}^{x_2+h} f(t)\,dt \leqslant 0,$$

which is impossible. Thus, $y_1 < y_2 < \cdots < y_{2b} < y_1 + 2\pi$ and sign $f(y_j) = (-1)^j$, $j = \overline{1, 2b}$. This gives $\nu(f) \geqslant 2b = \nu(f^h)$. $\square$

5. On existence of the spline from $S_h(S_n^r(\gamma, \delta))$ with prescribed minima

This section is devoted to the proof of Theorem 5. This theorem can be proved in many ways. We shall use methods from the paper [16].

Let $r, n = 1, 2, \ldots, 0 < h < \pi/n$ and $\gamma, \delta > 0$. Let $\tilde{N}_n^r$ denote the set of functions f which can be represented in the form

$$f = g^h + a, \quad g \in S_n^r(\gamma, \delta), \quad a \in \mathbb{R}$$

and have exactly $2n$ extrema on a period. It can be easily seen that the Steklov function of every $2\pi/n$-periodic function $f \in S_n^r(\gamma, \delta)$ belongs to the set $\tilde{N}_n^r$. Hence, $\tilde{N}_n^r \neq \emptyset$.

Let $f \in S_n^0(\gamma, \delta)$, and let

$$\xi_1 < \xi_2 < \cdots < \xi_{2n} < \xi_1 + 2\pi$$

be the nodes of the spline f such that $f(t) \equiv \gamma$ when $t \in (\xi_1, \xi_2)$. Then, since f has a zero mean value, the following equality holds

$$\sum_{j=1}^{2n} (-1)^j \xi_j = \frac{2\pi\delta}{\gamma + \delta}. \tag{5.1}$$

Hence, every system of points $\xi_1 < \xi_2 < \cdots < \xi_{2n-1} < \xi_1 + 2\pi$ such that

$$\frac{2\pi\delta}{\gamma + \delta} - \sum_{j=1}^{2n-1} (-1)^j \xi_j < \xi_1 + 2\pi$$

uniquely determines some spline $f \in S_n^0(\gamma, \delta)$. Such a system of points we shall denote by ξ ($\xi = \{\xi_j\}_{j=1}^{2n-1}$), and we shall call it as *the determining system* for the spline f. In addition, we shall denote by f_ξ the spline which corresponds to the system of points ξ. Let ξ be a given determining system for some spline. Then set

$$\xi_{2n} = \frac{2\pi\delta}{\gamma + \delta} - \sum_{j=1}^{2n-1} (-1)^j \xi_j$$

and

$$\xi_{2n+1} = \xi_1 + 2\pi.$$

Lemma 5.1. *Let ξ, η be determining systems for splines f_ξ and f_η, respectively. If the difference $f_\xi(t) - f_\eta(t)$ changes sign exactly $2n$ times on $[0, 2\pi)$ and*

$$\xi_j < \eta_{j+1} < \xi_{j+2}, \quad j = \overline{1, 2n-1},$$

then it is necessary that $\xi_j \neq \eta_j$ for every $j = \overline{1, 2n}$.

Proof. Let $\xi_0 = \xi_{2n} - 2\pi$ and $\eta_0 = \eta_{2n} - 2\pi$. It is easy to verify that the difference $f_\xi(t) - f_\eta(t)$ changes sign at most once on each of intervals (ξ_{j-1}, ξ_{j+1}) and (η_{j-1}, η_{j+1}), $j = \overline{1, 2n}$. Assume to the contrary, there exists $1 \leqslant j \leqslant 2n$ such that $\xi_j = \eta_j$. There are two possible cases: $\xi_{j-1} \leqslant \eta_{j-1}$ and $\xi_{j-1} \geqslant \eta_{j-1}$. We will consider the first case. In this case $f_\xi(t) - f_\eta(t)$ does not have sign changes on the interval (ξ_{j-1}, ξ_{j+1}), which contradicts the assumption $\nu(f_\xi - f_\eta) = 2n$. The second one can be studied similarly. $\square$

We shall denote by $U_\rho(\xi)$ the closed ball with the center $\xi = (\xi_1, \ldots, \xi_{2n-1})$ and the radius $\rho > 0$, in $(2n-1)$-dimensional space $\mathbb{R}^{2n-1}$ with the norm $\|\xi\| := \max\limits_j |\xi_j|$.

Lemma 5.2. *Let $\xi \in \mathbb{R}^{2n-1}$ be the determining system for the spline $f_\xi \in S_n^0(\gamma, \delta)$. Then there exists $\rho > 0$ such that an arbitrary point $\eta \in U_\rho(\xi)$ is a determining system for some spline $f_\eta \in S_n^0(\gamma, \delta)$.*

This lemma can be proved similarly to Lemma 3.2 in [16].

Let ξ be the determining system for the spline $f_\xi \in S_n^0(\gamma, \delta)$ such that $f_{\xi,r}^h \in \tilde{N}_n^r$, where

$$f_{\xi,r}(t) = (I_r f_\xi)(t) = (D_r * f_\xi)(t) := \int_0^{2\pi} D_r(t-\tau) f(\tau)\, d\tau.$$

Since I_r is a bounded operator, we may assume ρ to be such that $f_{\eta,r}^h \in \tilde{N}_n^r$ for every $\eta \in U_\rho(\xi)$. Let us consider an arbitrary interval $(a, a+2\pi)$ containing n points $x_1 < x_2 < \cdots < x_n$ at which $f_{\xi,r}^h$ attains its minima. We may choose $\rho > 0$ such that for every $\eta \in U_\rho(\xi)$ points $y_1 < y_2 < \cdots < y_n$ at which $f_{\eta,r}^h$ attains its minima, belong to the interval $(a, a+2\pi)$.

For every point $\eta \in U_\rho(\xi)$ let

$$\tau(\eta) = \{y_1, \ldots, y_n, f_{\eta,r}^h(y_2) - f_{\eta,r}^h(y_1), \ldots, f_{\eta,r}^h(y_n) - f_{\eta,r}^h(y_1)\}.$$

Clearly, the mapping τ from the ball $U_\rho(\xi)$ into $\mathbb{R}^{2n-1}$ is continuous.

Lemma 5.3. *There exists $\rho' < \rho$ such that the restriction of the mapping τ to the ball $U_{\rho'}(\xi)$ is injective.*

Proof. Let $a \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{2n} < a + 2\pi$ be the points at which $f_{\xi,r}^h$ attains its local extrema. Set $m_j := f_{\xi,r}^h(t_j)$, $j = \overline{1, 2n}$. Let us denote by w_0 the smallest number satisfying the equality

$$\omega(f_{\xi,r}^h; w) = \frac{1}{2} \min_{j=\overline{1,2n}} |m_{j+1} - m_j|,$$

where $m_{2n+1} = m_1$, and $\omega(g; t)$ is the modulus of continuity of the function g.

Let $\theta := \frac{1}{4} \min\limits_{j=\overline{1,2n}} |\xi_{j+1} - \xi_j|$. For every $0 < \varepsilon < \frac{1}{8} \min\limits_{j=\overline{1,2n}} |m_{j+1} - m_j|$ let us choose ρ' such that $\rho' < \min\{\rho; w_0/2; \theta/2\}$ and for an arbitrary $\eta \in U_{\rho'}(\xi)$ the distance between functions

$f^h_{\eta,r}$ and $f^h_{\xi,r}$ in the L_∞-metric does not exceed ε. Due to the definition of numbers ε and w_0, we have that the distance between the neighboring points of local extremum of the function $f^h_{\eta,r}$, $\eta \in U_{\rho'}(\xi)$, is greater than or equal to w_0.

Now suppose the assertion of the lemma is false. Then there exist two points $\eta, \zeta \in U_{\rho'}(\xi)$, $\eta \neq \zeta$, such that $\tau(\eta) = \tau(\zeta)$. Let $\{y_j\}_{j=1}^n$ and $\{z_j\}_{j=1}^n$ be the points from the interval $(a, a+2\pi)$ at which $f^h_{\eta,r}$ and $f^h_{\zeta,r}$ attain their local minima, respectively. Hence, $y_j = z_j$ and $f^h_{\eta,r}(y_j) - f^h_{\eta,r}(y_1) = f^h_{\zeta,r}(z_j) - f^h_{\zeta,r}(z_1)$, $j = \overline{1,n}$.

Let $u = \zeta_1 - \eta_1$, and let us consider the function $f_\eta(t-u)$. Let us consider the case $u > 0$ in detail. The case $u < 0$ can be studied similarly. Clearly, $\{\zeta_1, \eta_2+u, \ldots \eta_{2n-1}+u\}$ is a determining system for the spline $f_\eta(t-u)$. Let η_{2n} be chosen such that

$$\eta_{2n} = \frac{2\pi\delta}{\gamma+\delta} - \sum_{j=1}^{2n-1}(-1)^j \eta_j.$$

For every $j = \overline{1, 2n-2}$ we have

$$\eta_j + u < \eta_j + 2\theta \leqslant \zeta_{j+1} \leqslant \eta_{j+2} - 2\theta < \eta_{j+2} + u,$$

since $|u| \leqslant 2\rho' < 2\theta$.

Let us apply Lemma 5.1 to determining systems for the splines $f_\eta(t-u)$ and $f_\zeta(t)$. Since the first points of these systems are equal, Lemma 5.1 shows that the difference $f_\eta(t-u) - f_\zeta(t)$ has at most $(2n-1)$ sign changes. Define

$$g_1(t) := f^h_{\eta,r}(t-u) - f^h_{\eta,r}(y_1),$$

$$g_2(t) := f^h_{\zeta,r}(t) - f^h_{\zeta,r}(y_1).$$

We shall show that the difference $g_1(t) - g_2(t)$ has at least two sign changes on every interval $[y_j, y_{j+1}]$, $j = \overline{1, 2n}$. Since $|u| < 2\rho' < w_0$,

$$g_1(y_j) - g_2(y_j) > 0, \quad j = \overline{1,n}.$$

Furthermore,

$$g_1(y_j + u) - g_2(y_j + u) < 0, \quad j = \overline{1,n}.$$

Hence,

$$\nu(g_1' - g_2') \geqslant \nu(g_1 - g_2) \geqslant 2n.$$

At the same time

$$g_1'(t) - g_2'(t) = f^h_{\eta,r-1}(t-u) - f^h_{\zeta,r-1}(t)$$

and $\nu(f^h_{\eta,r-1}) = \nu(f^h_{\zeta,r-1}) = 2n$. Therefore, applying Rolle's theorem and Theorem 7 we obtain

$$\begin{aligned} 2n &\leqslant \nu(g_1 - g_2) \leqslant \nu(g_1' - g_2') = \nu(f^h_{\eta,r-1}(\cdot - u) - f^h_{\zeta,r-1}(\cdot)) \\ &\leqslant \nu(f^h_\eta(\cdot - u) - f^h_\zeta(\cdot)) \leqslant \nu(f_\eta(\cdot - u) - f_\zeta(\cdot)) \leqslant 2n-1, \end{aligned}$$

which is impossible. This proves the lemma. $\square$

Since the mapping τ is continuous, we derive from the last lemma that τ is a homeomorphism from $U_{\rho'}(\xi)$ into $\mathbb{R}^{2n-1}$.

Let us denote by E the set of points $x = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ such that $0 < x_1 < \dots < x_{n-1} < 2\pi$. Obviously, E is a connected set. Let $E_0^r \subset E$ be such that for every point $x \in E_0^r$ there exists a function $f_{\xi,r}^h \in \tilde{N}_n^r$ with equal local minima at the points $0, x_1, \dots, x_{n-1}$. The set E_0^r is non-empty, since

$$(2\pi/n, 4\pi/n, \dots, 2(n-1)\pi/n) \in E_0^r.$$

In fact, for an arbitrary $2\pi/n$-periodic function $f_{\xi,r}^h \in \tilde{N}_n^r$ we can choose a number b such that the function $f_{\xi,r}^h(t+b)$ attains its minima at the points $2k\pi/n$, $k = \overline{0, n-1}$.

Lemma 5.4. *The set E_0^r is open in E.*

Proof. For an arbitrary $x \in E_0^r$ there exists a function $f_{\xi,r}^h \in \tilde{N}_n^r$ that attains the equal local minima at the points $0, x_1, \dots, x_{n-1}$. Due to Lemma 5.3, there exists a ball $U_{\rho'}(\xi)$ such that the mapping $\tau(\eta) : U_{\rho'}(\xi) \to \mathbb{R}^{2n-1}$ is a homeomorphism. By virtue of theorem about invariance (see [1, p. 196]) of the domain, this provides the existence of an interior point $\xi \in \tau(U_{\rho'})$,

$$\tau(\xi) = (0, x_1, \dots, x_{n-1}, 0, \dots, 0) \in \tau(U_{\rho'}(\xi)).$$

Moreover, there exists a neighborhood of the point x such that for every point $y \in E$ from this neighborhood

$$(0, y_1, \dots, y_{n-1}, 0, \dots, 0) \in \tau(U_{\rho'}(\xi)).$$

Thus, there exists $\eta \in U_{\rho'}(\xi)$ such that $\tau(\eta) = (0, y_1, \dots, y_{n-1}, 0, \dots, 0)$. This completes the proof. $\square$

Lemma 5.5. *The set E_0^r is closed in E.*

Proof. Let $x \in E$ and let the sequence $\{x^m\}_{m=1}^{\infty} \subset E_0^r$ converges to x as $m \to \infty$. By definition of the sequence $\{x^m\}$, for every point x^m there exists a spline $f_{\xi^m} \in S_n^0(\gamma, \delta)$ with a determining system $\xi^m = \{\xi_j^m\}_{j=1}^{2n-1}$ such that

$$f_{\xi^m,r}^h(0) = f_{\xi^m,r}^h(x_j), \quad j = \overline{1, n-1}$$

and

$$f_{\xi^m,r-1}^h(0) = f_{\xi^m,r-1}^h(x_j^m) = 0, \quad j = \overline{1, n-1}.$$

It can be easily seen that there exists the subsequence $\{\xi^{m_k}\}$ which tends to some point $\xi \in \mathbb{R}^{2n-1}$ as $k \to \infty$. Clearly, ξ is the determining system for the spline $f_\xi \in S_n^0(\gamma, \delta)$. This implies that $\|f_{\xi^{m_k}} - f_\xi\|_1 \to 0$ as $k \to \infty$. From this, the sequence $\{f_{\xi^{m_k},b}^h\}$ converges uniformly to $f_{\xi,b}^h$ for an arbitrary integer $0 \leqslant b \leqslant r$. Thus, we have

$$f_{\xi^{m_k},r}^h(x_j^{m_k}) \to f_r^h(x_j) \quad \text{and} \quad f_{\xi^{m_k},r-1}^h(x_j^{m_k}) \to f_{r-1}^h(x_j),$$

as $k \to \infty$, for every $j = \overline{1, n-1}$, and

$$f^h_{\xi^{m_k},r}(0) \to f^h_r(0) \quad \text{and} \quad f^h_{\xi^{m_k},r-1}(0) \to f^h_{r-1}(0),$$

as $k \to \infty$. Hence,

$$f^h_r(x_j) = f^h_r(0), \quad j = \overline{1, n-1},$$

$$f^h_{r-1}(x_j) = f^h_{r-1}(0) = 0, \quad j = \overline{1, n-1},$$

and f^h_r attains its minima at the points $0, x_1, \ldots, x_{n-1}$. This proves the lemma. $\square$

To summarize, observe that E^r_0 is non-empty, open and closed subset in the connected set E. This gives $E^r_0 = E$. Thus, the last remark proves Theorem 5.

6. Proof of Theorem 6

In this section we shall prove the following:

Theorem 13. *Let $n, r = 1, 2, \ldots$, $0 < h < \pi/n$ and $\alpha, \beta, \gamma, \delta, \varepsilon > 0$. Then for every function $f \in S^r_n(\gamma, \delta)$,*

$$E_0(A_\varepsilon * \phi^h_{n,r;\gamma,\delta})_{1;\alpha,\beta} \leqslant E_0(A_\varepsilon * f^h)_{1;\alpha,\beta}.$$

We shall establish Theorem 6 by letting $\varepsilon \to 0$.

Let $n, r = 1, 2, \ldots$, $\gamma, \delta > 0$ and $0 < h < \pi/n$. Note that the nodes $x_1 < x_2 < \cdots < x_{2l} < x_1 + 2\pi$, $l \leqslant n$, of the spline $g \in S^r_n(\gamma, \delta)$ for which $g^{(r)}$ attains the value γ on the interval (x_1, x_2) satisfy

$$\sum_{j=1}^{2l} (-1)^j x_j = \frac{2\delta\pi}{\gamma + \delta}.$$

Fix $\alpha, \beta, \gamma, \delta, \varepsilon, n, r$ and consider the extremal problem

$$E_0(A_\varepsilon * g^h)_{1;\alpha,\beta} \to \inf, \quad g \in S^r_n(\gamma, \delta). \tag{6.1}$$

Since $A_\varepsilon * S_h(S^r_n(\gamma, \delta)) := \{A_\varepsilon * s \ : \ s \in S_h(S^r_n(\gamma, \delta))\}$ is compact in the topology of the uniform convergence and $E_0(A_\varepsilon * g^h)_{1;\alpha,\beta}$ continuously depends on $g \in S^r_n(\gamma, \delta)$, the solution of the problem (6.1) exists. Assume that the spline solving the problem (6.1) has exactly $2l$, $l \leqslant n$, nodes. Due to Lemmas 3.1 and 3.2 the nodes $x_1 < \cdots < x_{2l} < x_1 + 2\pi$ of this spline are also solutions of the following problem:

$$\frac{\alpha + \beta}{2} \int_0^{2\pi} \left| (\gamma + \delta) \sum_{j=1}^{2l} (-1)^j \int_0^{2\pi} D_1(x_j - t)[A_\varepsilon * D_r]^h(x - t)\, dt - \lambda \right| dx$$
$$+ 2\pi\lambda \frac{\beta - \alpha}{2} \to \min, \tag{6.2}$$

under the constraint

$$\sum_{j=1}^{2l} (-1)^j x_j = \frac{2\pi\delta}{\gamma + \delta}, \quad \lambda \in \mathbb{R}. \tag{6.3}$$

Due to Lemma 3.3, we can apply the Lagrange multiplier method to study problem (6.2). This implies the following necessary conditions to be satisfied by the solutions $x_1, \ldots, x_{2l}, \lambda$ of this problem:

$$-\frac{\alpha+\beta}{2}\int_0^{2\pi}\operatorname{sign}\left((\gamma+\delta)\sum_{j=1}^{2l}(-1)^j\int_0^{2\pi}D_1(x_j-t)[A_\varepsilon * D_r]^h(x-t)\,dt-\lambda\right)dx$$
$$+2\pi\frac{\beta-\alpha}{2}=0, \tag{6.4}$$

$$(-1)^k\frac{\alpha+\beta}{2}\cdot(\gamma+\delta)\int_0^{2\pi}[A_\varepsilon * D_r]^h(x_k-x)$$
$$\times\operatorname{sign}\left((\gamma+\delta)\sum_{j=1}^{2l}(-1)^j\int_0^{2\pi}D_1(x_j-t)[A_\varepsilon * D_r]^h(x-t)-\lambda\right)dx$$
$$=(-1)^{k+1}\Lambda,\quad k=\overline{1,2l}, \tag{6.5}$$

$$\sum_{j=1}^{2l}(-1)^j x_j=\frac{2\pi\delta}{\gamma+\delta}, \tag{6.6}$$

where Λ is the Lagrange multiplier.

Let $x_1 < x_2 < \cdots < x_{2l} < x_1 + 2\pi$ be such that the relation (6.6) holds. For a given number $m = 0, 1, \ldots$ set

$$f_m(x)=(\gamma+\delta)\sum_{j=1}^{2l}(-1)^{j+m}D_{m+1}(x_j-x).$$

Using this notation we have

$$(\gamma+\delta)\sum_{j=1}^{2l}(-1)^j\int_0^{2\pi}D_1(x_j-t)[A_\varepsilon * D_r]^h(x-t)\,dt=(A_\varepsilon * f_r^h)(x).$$

Conditions (6.4)–(6.6) can be written as follows. If $(x_1, \ldots, x_{2l}, \lambda)$ is a solution of the problem (6.2), then

(1) $f_r \in S_l^r(\gamma, \delta)$ and $A_\varepsilon * f_r^h \in A_\varepsilon * S_h(S_l^r(\gamma, \delta)$ so that $A_\varepsilon * f_r^h$ is a solution of the problem (6.1).
(2) λ is the constant of the best (α, β)-approximation of $A_\varepsilon * f_r^h$ in the space L_1 and if

$$g_0(x)=\alpha\operatorname{sign}((A_\varepsilon * f_r^h)(x)-\lambda)_+-\beta\operatorname{sign}((A_\varepsilon * f_r^h)(x)-\lambda)_-$$
$$=\frac{\alpha+\beta}{2}\operatorname{sign}((A_\varepsilon * f_r^h)(x)-\lambda)-\frac{\beta-\alpha}{2},$$

then

$$\operatorname{sign} g_0(x)=\operatorname{sign}((A_\varepsilon * f_r^h)(x)-\lambda) \tag{6.7}$$

and

$$g_r(x) = (D_r * g_0)(x) \in S_l^r(\alpha, \beta)$$

and consequently

$$(A_\varepsilon * g_r^h)(x) \in A_\varepsilon * S_h(S_l^r(\alpha, \beta)).$$

(3) $A_\varepsilon * g_r^h$ attains at the points x_j (nodes of f_0) the equal values and

$$\operatorname{sign}((A_\varepsilon * g_r^h)(x) - (A_\varepsilon * g_r^h)(x_1)) = \pm \operatorname{sign} f_0(x).$$

Note that the condition (1) follows from the relation (6.6). As for condition (2), the statement that λ is the constant of the best (α, β)-approximation of $A_\varepsilon * f_r^h$ in the space L_1 follows from condition (6.4) and Theorem 8. From the fact that $\nu(f_0) = 2l$, Lemma 4.1, Rolle's theorem, property (3.1) and relation (6.7) we have $g_r \in S_l^r(\gamma, \delta)$. Finally, as for condition (3), the fact that $(A_\varepsilon * g_r^h)(x)$ attains at the points x_j (nodes of f_0) equal values follows from condition (6.5). In addition, we can apply Lemma 4.1, Rolle's theorem and property (3.1) to verify that the difference

$$(A_\varepsilon * g_r^h)(x) - (A_\varepsilon * g_r^h)(x_1)$$

does not have zeros different from x_j and that this difference changes its sign at the points x_j.

We shall prove now the following:

Theorem 14. *Conditions* (1)–(3) *can be satisfied* (*up to a translation of the argument*) *only by the function* $(A_\varepsilon * f_r^h)(x) = (A_\varepsilon * \phi_{l,r;\gamma,\delta}^h)(x)$.

For a number $y \in \mathbb{R}$ set

$$F_{y,0}(x) := f_0(x) - f_0(x + y), \quad F_{y,r} = f_r(x) - f_r(x + y)$$

and

$$H_{y,0}(x) = g_0(x) - g_0(x + y), \quad H_{y,r}(x) = g_r(x) - g_r(x + y).$$

Function $A_\varepsilon * H_{y,r}^h$ has only isolated zeros. By $\mu(\varrho)$ let us denote the number of zeros of the function ϱ on a period counted according to the following rule: the simple isolated zeros of ϱ are counted once, while the multiple zeros are counted two times.

Lemma 6.1. *For any* $y \in \mathbb{R}$,

$$\nu(A_\varepsilon * H_{y,r}^h) \geqslant \nu(F_{y,0}).$$

Proof. In fact, if on a period there exist $2s$ points $t_1 < t_2 < \cdots < t_{2s}$ at which $F_{y,0}$ has non-zero values with alternating sign, then, by condition (3), $A_\varepsilon * H_{y,r}^h$ also has non-zero values at this points with alternating sign. This completes the proof. □

Lemma 6.2. *Let* $r \geqslant 2$. *Then for any* $y \in \mathbb{R}$,

$$\mu(A_\varepsilon * H_{y,r}^h) \leqslant \nu(F_{y,0}).$$

Proof. Lemma 6.2 is an analogue of Lemma 5.5 from the paper of Babenko [5]. Let $\mu(A_\varepsilon * H_{y,r}^h) = 2s$. Then by virtue of Rolle's theorem and our method of enumerating zeros on a period, there exist $2s$ different zeros for the function $(A_\varepsilon * H^h_{y,r})'$. However, between neighboring zeros of $(A_\varepsilon * H^h_{y,r})'$, the function $(A_\varepsilon * H^h_{y,r})''$ alternates its sign at least once. Applying Rolle's theorem we obtain

$$\nu(A_\varepsilon * H^h_{y,0}) \geqslant \cdots \geqslant \nu(A_\varepsilon * H^h_{y,r-2}) = \nu((A_\varepsilon * H^h_{y,r})'') \geqslant 2s.$$

From property (3.1) of the function $A_\varepsilon(x)$ we conclude that

$$\nu(H^h_{y,0}) \geqslant 2s.$$

Let us ensure that functions $g_0(x)$ and $g_0(x+y)$ satisfy conditions of Theorem 7. To this end it is suffices to verify that the function g^h_0 has exactly $2l$ sign changes on a period. By condition (3), the function

$$(A_\varepsilon * g^h_r)(x) - (A_\varepsilon * g^h_r)(x_1)$$

changes its sign at nodes of f_0. This implies that this function has exactly $2l$ sign changes. Hence, due to property (3.1) the difference

$$g^h_r(x) - g^h_r(x_1)$$

has at least $2l$ sign changes on a period. However, by Rolle's theorem,

$$\nu(g^h_0) \geqslant 2l.$$

Finally, by Lemma 4.1, the function g_0 has at least $2l$ sign changes. At the same time, due to condition (2), $g_0 \in S^r_l(\gamma, \delta)$. This provides that g_0 and consequently g^h_0 have exactly $2l$ sign changes on a period.

Thus, functions $g_0(x)$ and $g_0(x+y)$ satisfy conditions of Theorem 7. Applying Theorem 7 we conclude that

$$\nu(H_{y,0}) \geqslant 2s.$$

As a consequence, there exist $2s$ points $t_1, \ldots, t_{2s}$ on a period such that $H_{y,0}$ attains non-zero values at these points and alternates its sign when an argument passing from t_j to t_{j+1}. Because of (6.7), we have

$$\nu(A_\varepsilon * F^h_{y,r}) \geqslant 2s.$$

Applying Rolle's theorem and property (3.1) yields

$$\mu(A_\varepsilon * H^h_{y,r}) = 2s \leqslant \nu(A_\varepsilon * F^h_{y,r}) \leqslant \nu(A_\varepsilon * F^h_{y,0}) \leqslant \nu(F^h_{y,0}). \tag{6.8}$$

Functions $f_0(x)$ and $f_0(x+y)$ satisfy conditions of Theorem 7. In fact, we have already established that

$$\nu(g_0) = 2l.$$

By relation (6.7)

$$\nu((A_\varepsilon * f^h_r)(\cdot) - \lambda) = 2l.$$

Hence, applying property (3.1) and Rolle's theorem we obtain

$$\nu(f^h_0) \geqslant 2l.$$

However, $\nu(f_0^h)\leqslant\nu(f_0)$ (Lemma 4.1), and since $\nu(f_0)=2l$ from the definition of f_0 we conclude that

$$\nu(f_0^h)=2l.$$

Finally, applying Theorem 7 yields

$$\nu(F_{y,0}^h)\leqslant\nu(F_{y,0}).$$

Comparing (6.8) with the latter inequality we obtain

$$\mu(A_\varepsilon * H_{y,r}^h)\leqslant\nu(F_{y,0}). \qquad \square$$

Proof of Theorem 14. Due to Lemmas 6.1 and 6.2 we conclude that

$$\mu(A_\varepsilon * H_{y,r}^h)=\nu(A_\varepsilon * H_{y,r}^h)=\nu(F_{y,0})$$

as $\mu(A_\varepsilon * H_{y,r}^h)\geqslant\nu(A_\varepsilon * H_{y,r}^h)$ for any $y\in\mathbb{R}$ for which $A_\varepsilon * H_{y,r}^h$ and $F_{y,0}$ are not identically zero. Thus, every non-identically zero difference must have only isolated simple zeros. We shall show that it follows the function $A_\varepsilon * g_r^h$ is $2\pi n^{-1}$-periodic.

Let T be the minimal period for $A_\varepsilon * g_r^h$ and let a_1 be the point of the smallest local maximum of $A_\varepsilon * g_r^h$. We prove that $A_\varepsilon * g_r^h$ has exactly two zeros on the interval $[a_1, a_1+T)$. Assume to the contrary that the function $A_\varepsilon * g_r^h$ has at least four zeros on the interval $[a_1, a_1+T)$. However, then there is at least one local maximum of $A_\varepsilon * g_r^h$ on this interval. Let a_2 be the point of local maximum of $A_\varepsilon * g_r^h$ nearest to a_1 from the right, and a_3 the local maximum of $A_\varepsilon * g_r^h$ nearest to a_1+T from the left. Moreover, let b_1 be the point of local minimum of $A_\varepsilon * g_r^h$ nearest to a_1 from the right, and b_2 the local minimum of $A_\varepsilon * g_r^h$ nearest to a_1+T from the left. We shall prove that there exists $y\in(0,T)$ such that $A_\varepsilon * H_{y,r}^h$ has a multiple zero at some point on the period. This will show that $A_\varepsilon * g_r^h$ has a period $y<T$, i.e., we obtain a contradiction to the minimality of the period T.

If $(A_\varepsilon * g_r^h)(a_1)=(A_\varepsilon * g_r^h)(a_2)$, then we can choose $y=a_2-a_1<T$. Hence, $(A_\varepsilon * H_{y,r}^h)(a_1)=(A_\varepsilon * g_r^h)(a_1)-(A_\varepsilon * g_r^h)(a_1+a_2-a_1)=0$ and $(A_\varepsilon * H_{y,r}^h)'(a_1)=0$. This provides a_1 is a multiple zero of $A_\varepsilon * H_{y,r}^h$. Now assume $(A_\varepsilon * g_r^h)(a_2)>(A_\varepsilon * g_r^h)(a_1)$ and $(A_\varepsilon * g_r^h)(a_3)>(A_\varepsilon * g_r^h)(a_1)$. Let us consider the values $(A_\varepsilon * g_r^h)(b_1)$ and $(A_\varepsilon * g_r^h)(b_2)$. If they are equal, then we can choose $y=b_2-b_1$. Without loss of generality, we may assume $(A_\varepsilon * g_r^h)(b_2)>(A_\varepsilon * g_r^h)(b_1)$. Hence, there exist $c_1\in(a_1,b_1)$ and $c_2\in(a_3,b_2)$ such that $(A_\varepsilon * g_r^h)(c_1)=(A_\varepsilon * g_r^h)(b_2)$ and $(A_\varepsilon * g_r^h)(c_2)=(A_\varepsilon * g_r^h)(a_1)$. Let us show that there exist $\xi\in[a_1,c_1]$ and $\eta\in[c_2,b_2]$ such that $(A_\varepsilon * g_r^h)(\xi)=(A_\varepsilon * g_r^h)(\eta)$ and $(A_\varepsilon * g_r^h)'(\xi)=(A_\varepsilon * g_r^h)'(\eta)$.

It can be easily seen that $A_\varepsilon * g_r^h$ decreases on the intervals $[a_1,c_1]$ and $[c_2,b_2]$. In addition, $(A_\varepsilon * g_r^h)(t)$ attains every value from the interval $[(A_\varepsilon * g_r^h)(b_2),(A_\varepsilon * g_r^h)(a_1)]$, when $t\in[a_1,c_1]$. Similarly, $(A_\varepsilon * g_r^h)(t)$ attains every value from the interval $[(A_\varepsilon * g_r^h)(b_2),(A_\varepsilon * g_r^h)(a_1)]$, when $t\in[c_2,b_2]$. Therefore, there exist functions $\psi_1=(A_\varepsilon * g_r^h|_{[a_1,c_1]})^{-1}$ and $\psi_2=(A_\varepsilon * g_r^h|_{[c_2,b_2]})^{-1}$, defined on the interval $[(A_\varepsilon * g_r^h)(b_2),(A_\varepsilon * g_r^h)(a_1)]$, which are continuously differentiable. Then $\lim_{x\to x_0}\psi_1'(x)=\infty$, when $x_0=(A_\varepsilon * g_r^h)(a_1)$, and is finite, when $x_0=(A_\varepsilon * g_r^h)(b_2)$. In addition, $\lim_{x\to x_0}\psi_2'(x)=\infty$, when $x_0=(A_\varepsilon * g_r^h)(b_2)$, and is finite, when $x_0=(A_\varepsilon * g_r^h)(a_1)$. Thus, there exists $w\in[(A_\varepsilon * g_r^h)(b_2),(A_\varepsilon * g_r^h)(a_1)]$ such that $\psi_1'(w)=\psi_2'(w)$. Hence, there exist $\xi\in(a_1,c_1)$ and $\eta\in(c_2,b_2)$ such that $(A_\varepsilon * g_r^h)(\xi)=(A_\varepsilon * g_r^h)(\eta)=w$ and

$$(A_\varepsilon * g_r^h)'(\xi)=\frac{1}{\psi_1'(w)}=\frac{1}{\psi_2'(w)}=(A_\varepsilon * g_r^h)'(\eta).$$

Then $y = \eta - \xi < T$ is a period of $A_\varepsilon * g_r^h$, which is impossible. This implies $A_\varepsilon * g_r^h$ has exactly two zeros on $[a_1, a_1 + T)$. Since $A_\varepsilon * g_r^h$ has $2l$ zeros on $[0, 2\pi)$, from the last note we have that $T = 2\pi l^{-1}$. As a consequence $A_\varepsilon * g_r^h$ has period $2\pi l^{-1}$. However, then both f_0 and $A_\varepsilon * f_r^h$ are $2\pi l^{-1}$-periodic, so that $A_\varepsilon * f_r^h = A_\varepsilon * \phi^h_{l,r;\gamma,\delta}$ up to a translation of the argument. Theorem 14 is proved. □

To prove Theorem 13 it remains to show that

$$E_0(A_\varepsilon * \phi^h_{n,r;\gamma,\delta})_{1;\alpha,\beta} < E_0(A_\varepsilon * \phi^h_{l,r;\gamma,\delta})_{1;\alpha,\beta} \tag{6.9}$$

as soon as $l < n$.

The proof falls naturally into four parts.

Lemma 6.3. *Let $l < n$, $\gamma, \delta > 0$ and $r = 1, 2, \ldots$. Then for an arbitrary $x \in [0, 2\pi)$,*

$$\min_t (A_\varepsilon * \phi^h_{l,r;\gamma,\delta})(t) < (A_\varepsilon * \phi^h_{n,r;\gamma,\delta})(x) < \max_t (A_\varepsilon * \phi^h_{l,r;\gamma,\delta})(t). \tag{6.10}$$

Proof. We shall prove the second inequality of (6.10). The first one can be established similarly.

From Lemma 3.4 we have that $(A_\varepsilon * \phi_{n,r;\gamma,\delta})(x) < \max_t (A_\varepsilon * \phi_{l,r;\gamma,\delta})(t)$ for an arbitrary $x \in [0, 2\pi)$. Let $y, z \in \mathbb{R}$ be such that

$$\max_t (A_\varepsilon * \phi^h_{n,r;\gamma,\delta})(t) = (A_\varepsilon * \phi^h_{n,r;\gamma,\delta})(y)$$

and

$$\max_t (A_\varepsilon * \phi^h_{l,r;\gamma,\delta})(t) = (A_\varepsilon * \phi^h_{l,r;\gamma,\delta})(z).$$

Let us consider the function $f(t) = (A_\varepsilon * \phi_{l,r;\gamma,\delta})(z + t) - (A_\varepsilon * \phi_{n,r;\gamma,\delta})(y + t)$. It follows that $f(-h) = f(h)$, and there exists a point $\xi \in [-h, h]$ such that $f(\xi) > 0$. It can be easily seen that f does not have sign changes on $[-h, h]$ when $f(h) > 0$. Then $f(t) > 0$ for every $t \in [-h, h]$ and

$$f^h(0) = (A_\varepsilon * \phi^h_{l,r;\gamma,\delta})(z) - (A_\varepsilon * \phi^h_{n,r;\gamma,\delta})(y) = \frac{1}{2h} \int_{-h}^{h} f(t)\, dt > 0.$$

Now we shall consider the case $f(h) < 0$. Let the point $\mu_0 \in (-h - 2\pi/n, -h)$ be such that $(A_\varepsilon * \phi_{l,r;\gamma,\delta})(\mu_0) = (A_\varepsilon * \phi_{l,r;\gamma,\delta})(\mu_0 + 2\pi/n)$. Then f has exactly two sign changes on the interval $[\mu_0, \mu_0 + 2\pi/n]$. Therefore,

$$\begin{aligned} f^h(0) &= (A_\varepsilon * \phi^h_{l,r;\gamma,\delta})(z) - (A_\varepsilon * \phi^h_{n,r;\gamma,\delta})(y) \\ &= \frac{1}{2h} \int_{-h}^{h} f(t)\, dt \geqslant \frac{1}{2h} \int_{\mu}^{\mu+2\pi/n} f(t)\, dt > 0, \end{aligned}$$

which can be easily verified. This completes the proof. □

Lemma 6.4. *Let $\xi, \eta \in \mathbb{R}$ be such that $(A_\varepsilon * \phi^h_{n,r;\gamma,\delta})(\xi) = (A_\varepsilon * \phi^h_{l,r;\gamma,\delta})(\eta)$. Then*

$$|(A_\varepsilon * \phi^h_{n,r-1;\gamma,\delta})(\xi)| \leqslant |(A_\varepsilon * \phi^h_{l,r-1;\gamma,\delta})(\eta)|$$

as soon as $(A_\varepsilon * \phi^h_{n,r-1;\gamma,\delta})(\xi) \cdot (A_\varepsilon * \phi^h_{l,r-1;\gamma,\delta})(\eta) > 0$.

Proof. Let $x_1 < x_2 < \cdots < x_{2l} < x_1 + 2\pi$ be the points of extrema of the function $A_\varepsilon * \phi^h_{l,r;\gamma,\delta}$. Assume to the contrary, that there exist points $\xi, \eta \in \mathbb{R}$ such that

$$(A_\varepsilon * \phi^h_{n,r;\gamma,\delta})(\xi) = (A_\varepsilon * \phi^h_{l,r;\gamma,\delta})(\eta)$$

and

$$|(A_\varepsilon * \phi^h_{n,r-1;\gamma,\delta})(\xi)| > |(A_\varepsilon * \phi^h_{l,r-1;\gamma,\delta})(\eta)|,$$

although $(A_\varepsilon * \phi^h_{n,r-1;\gamma,\delta})(\xi) \cdot (A_\varepsilon * \phi^h_{l,r-1;\gamma,\delta})(\eta) > 0$. Applying Theorem 7 we obtain that the function $f(t) = (A_\varepsilon * \phi^h_{l,r;\gamma,\delta})(t) - (A_\varepsilon * \phi^h_{n,r;\gamma,\delta})(t + \xi - \eta)$ has exactly one zero on every interval $[x_j, x_{j+1})$, $j = \overline{1, 2l}$, $x_{2l+1} = x_1 + 2\pi$. Without loss of generality we may assume that $(A_\varepsilon * \phi^h_{n,r-1;\gamma,\delta})(\xi) > 0$. This implies $f(\eta) = 0$ and $f'(\eta) < 0$. Let $\eta \in [x_j, x_{j+1})$. Thus, there exists at least one zero of f either on the interval (η, x_{j+1}) or on the interval $[x_j, \eta)$, which is impossible. □

Lemma 6.5. *Let $l < n$. Then*

$$(A_\varepsilon * \phi^h_{n,r;\gamma,\delta})_\pm \prec (A_\varepsilon * \phi^h_{l,r;\gamma,\delta})_\pm. \tag{6.11}$$

Proof. Let us consider the rearrangements of the functions $(A_\varepsilon * \phi^h_{l,r;\gamma,\delta})(t) - \lambda$ and $(A_\varepsilon * \phi^h_{n,r;\gamma,\delta})(t) - \lambda$ for an arbitrary $\lambda \in \mathbb{R}$. Applying Lemma 6.3, yields

$$\Pi(A_\varepsilon * \phi^h_{n,r;\gamma,\delta} - \lambda, 0) < \Pi(A_\varepsilon * \phi^h_{l,r;\gamma,\delta} - \lambda, 0) \quad \text{and}$$

$$\Pi(A_\varepsilon * \phi^h_{n,r;\gamma,\delta} - \lambda, 2\pi) > \Pi(A_\varepsilon * \phi^h_{l,r;\gamma,\delta} - \lambda, 2\pi).$$

Obviously,

$$\int_0^{2\pi} \Pi(A_\varepsilon * \phi^h_{n,r;\gamma,\delta} - \lambda, t)\, dt = \int_0^{2\pi} \Pi(A_\varepsilon * \phi^h_{l,r;\gamma,\delta} - \lambda, t) = -2\pi\lambda. \tag{6.12}$$

It follows that $\Pi(A_\varepsilon * \phi^h_{n,r;\gamma,\delta} - \lambda, t)$ and $\Pi(A_\varepsilon * \phi^h_{l,r;\gamma,\delta} - \lambda, t)$ intersect at least at one point on $[0, 2\pi)$. We shall prove that there exists exactly one point of intersection of these functions. Assume to the contrary that there exist two points of intersection of $\Pi(A_\varepsilon * \phi^h_{n,r;\gamma,\delta} - \lambda, t)$ and $\Pi(A_\varepsilon * \phi^h_{l,r;\gamma,\delta} - \lambda, t)$. Hence, there exist points x_n and x_l such that

$$\Pi(A_\varepsilon * \phi^h_{n,r;\gamma,\delta} - \lambda, x_n) = \Pi(A_\varepsilon * \phi^h_{l,r;\gamma,\delta} - \lambda, x_l) = z$$

and $\Pi'(A_\varepsilon * \phi^h_{n,r;\gamma,\delta} - \lambda, x_n) < \Pi'(A_\varepsilon * \phi^h_{l,r;\gamma,\delta} - \lambda, x_l)$.

Let points $x'_n < x''_n$ and $x'_l < x''_l$ from $[0, 2\pi)$ be such that

$$(A_\varepsilon * \phi^h_{n,r;\gamma,\delta})(x'_n) = (A_\varepsilon * \phi^h_{n,r;\gamma,\delta})(x''_n) = (A_\varepsilon * \phi^h_{l,r;\gamma,\delta})(x'_l) = (A_\varepsilon * \phi^h_{l,r;\gamma,\delta})(x''_l) = z$$

and $(A_\varepsilon * \phi^h_{n,r;\gamma,\delta})(x) > z$ for every $x \in (x'_n, x''_n)$ as well as $(A_\varepsilon * \phi^h_{l,r;\gamma,\delta})(x) > z$ for every $x \in (x'_l, x''_l)$, since the equality $(A_\varepsilon * \phi^h_{n,r;\gamma,\delta})(x) = c$, $c \in (\min_u (A_\varepsilon * \phi^h_{n,r;\gamma,\delta})(u),$

$\max_u(A_\varepsilon * \phi^h_{n,r;\gamma,\delta})(u))$, always has exactly $2n$ solutions on the period. Thus,

$$\Pi'(A_\varepsilon * \phi^h_{n,r;\gamma,\delta} - \lambda, x_n) = \frac{1}{n} \cdot \frac{1}{\dfrac{1}{(A_\varepsilon * \phi^h_{n,r-1;\gamma,\delta})(x''_n)} - \dfrac{1}{(A_\varepsilon * \phi^h_{n,r-1;\gamma,\delta})(x'_n)}}$$

and

$$\Pi'(A_\varepsilon * \phi^h_{l,r;\gamma,\delta} - \lambda, x_l) = \frac{1}{l} \cdot \frac{1}{\dfrac{1}{(A_\varepsilon * \phi^h_{l,r-1;\gamma,\delta})(x''_l)} - \dfrac{1}{(A_\varepsilon * \phi^h_{l,r-1;\gamma,\delta})(x'_l)}}.$$

Applying Lemma 6.4 we obtain

$$(A_\varepsilon * \phi^h_{n,r-1;\gamma,\delta})(x'_n) < (A_\varepsilon * \phi^h_{l,r-1;\gamma,\delta})(x'_l) \quad \text{and}$$
$$(A_\varepsilon * \phi^h_{n,r-1;\gamma,\delta})(x''_n) > (A_\varepsilon * \phi^h_{l,r-1;\gamma,\delta})(x''_l).$$

This provides

$$\begin{aligned}\Pi'(A_\varepsilon * \phi^h_{l,r;\gamma,\delta} - \lambda, x_l) &= \frac{1}{l} \cdot \frac{1}{\dfrac{1}{(A_\varepsilon * \phi^h_{l,r-1;\gamma,\delta})(x''_l)} - \dfrac{1}{(A_\varepsilon * \phi^h_{l,r-1;\gamma,\delta})(x'_l)}} \\ &\leqslant \frac{1}{l} \cdot \frac{1}{\dfrac{1}{(A_\varepsilon * \phi^h_{n,r-1;\gamma,\delta})(x''_n)} - \dfrac{1}{(A_\varepsilon * \phi^h_{n,r-1;\gamma,\delta})(x'_n)}} \\ &\leqslant \Pi'(A_\varepsilon * \phi^h_{n,r;\gamma,\delta} - \lambda, x_n),\end{aligned}$$

which is impossible. Therefore, for every $x \in [0, 2\pi)$

$$\int_0^x \Pi(A_\varepsilon * \phi^h_{l,r;\gamma,\delta} - \tilde{\lambda}, t)\, dt \geqslant \int_0^x \Pi(A_\varepsilon * \phi^h_{n,r;\gamma,\delta} - \tilde{\lambda}, t)\, dt,$$

where $\tilde{\lambda} = \Pi(A_\varepsilon * \phi^h_{l,r;\gamma,\delta}, 2\pi)$. Due to (6.12), it follows immediately that inequality (6.11) holds for arbitrary $\lambda \in \mathbb{R}$, which is the desired conclusion. □

Relation (6.9) easily follows from Lemma 6.5. In fact, taking $x = 2\pi$ and λ, to be the constant of the best (α, β)-approximation of the function $A_\varepsilon * \phi^h_{l,r;\gamma,\delta}$ in the space L_1, we can assert that

$$\begin{aligned}&E_0(A_\varepsilon * \phi^h_{n,r;\gamma,\delta})_{1;\alpha,\beta} \\ &\leqslant \int_0^{2\pi} [\alpha((A_\varepsilon * \phi^h_{n,r;\gamma,\delta})(t) - \lambda)_+ + \beta((A_\varepsilon * \phi^h_{n,r;\gamma,\delta})(t) - \lambda)_-]\, dt \\ &= \alpha \int_0^{2\pi} P((A_\varepsilon * \phi^h_{n,r;\gamma,\delta})_+, t)\, dt + \beta \int_0^{2\pi} P((A_\varepsilon * \phi^h_{n,r;\gamma,\delta})_-, t)\, dt \\ &\leqslant \alpha \int_0^{2\pi} P((A_\varepsilon * \phi^h_{l,r;\gamma,\delta})_+, t)\, dt + \beta \int_0^{2\pi} P((A_\varepsilon * \phi^h_{l,r;\gamma,\delta})_-, t)\, dt \\ &= E_0(A_\varepsilon * \phi^h_{l,r;\gamma,\delta})_{1;\alpha,\beta}.\end{aligned}$$

Thus, the inequality (6.8) holds, which proves Theorem 13. Letting $\varepsilon \to 0$, we obtain that Theorem 6 holds.

7. Optimal interval quadrature formula on classes $W^r F$ (Proof of Theorems 1–4)

Let $n, r = 1, 2, \ldots$, $0 < h < \pi/n$ and $\gamma, \delta > 0$. Let $x_1 < x_2 < \cdots < x_n < x_1 + 2\pi$. Due to Theorem 5, there exists the spline $f^h_{\pm,\bar{x};\gamma,\delta} \in S_h(S^r_n(\gamma,\delta))$ such that it attains equal minimal values at the points $\{x_j\}_{j=1}^n$. Then,

$$\inf_{a_j} \sup_{f \in W^r_{\infty;\gamma^{-1},\delta^{-1}}} \left[\pm \int_0^{2\pi} f^h(t)\,dt \mp \sum_{j=1}^n a_j f^h(x_j) \right]$$

$$\geqslant \int_0^{2\pi} [\pm f^h_{\pm,\bar{x};\gamma,\delta}(t) - \min_u (\pm f^h_{\pm,\bar{x};\gamma,\delta}(u))]\,dt. \tag{7.1}$$

For the formula κ^i_n with equidistant nodes we have

$$R^{\pm}(W^r_{\infty;\gamma^{-1},\delta^{-1}}, \kappa^i_n) = R^{\pm}(S_h(W^r_{\infty;\gamma^{-1},\delta^{-1}}), \kappa_n)$$

$$= \int_0^{2\pi} [\pm \phi^h_{n,r;\gamma,\delta}(t) - \min_u (\pm \phi^h_{n,r;\gamma,\delta}(u))]\,dt. \tag{7.2}$$

In fact, due to (7.1), it suffices to prove that the left-hand side does not exceed the right-hand side. Let λ be the constant of best (γ,δ)-approximation of $S_h(m_{n,r})$. Restricting our consideration to $R^+(W^r_{\infty;\gamma^{-1},\delta^{-1}}, \kappa^i_n)$ and taking into account (1.4) and Theorem 9, we have

$$R^+(W^r_{\infty;\gamma^{-1},\delta^{-1}}, \kappa^i_n)$$

$$= R^+(S_h(W^r_{\infty;\gamma^{-1},\delta^{-1}}), \kappa_n) = E_0(S_h(m_{n,r}))_{1;\gamma,\delta}$$

$$= -\int_0^{2\pi} \frac{2\pi}{n} \sum_{j=1}^n D^h_r\left(\frac{2j\pi}{n} - x\right) [\gamma \operatorname{sign}(S_h(m_{n,r}) - \lambda)_+ - \delta \operatorname{sign}(S_h(m_{n,r}) - \lambda)_-]\,dx$$

$$\leqslant -\frac{2\pi}{n} \cdot n \cdot \min_t \phi^h_{n,r;\gamma,\delta}(t) = \int_0^{2\pi} [\phi^h_{n,r;\gamma,\delta}(x) - \min_t \phi^h_{n,r;\gamma,\delta}(t)]\,dx.$$

Finally, note that from Theorem 6 the equality

$$\inf_{g \in S^r_n(\gamma,\delta)} E_0^{\pm}(g^h)_1 = E_0^{\pm}(\phi^h_{n,r;\gamma,\delta})_1 \tag{7.3}$$

easily follows.

Comparing relations (7.1)–(7.3), we conclude that Theorem 4 holds.

Now we are ready to prove Theorem 2. In view of Theorem 10, it suffices to prove Theorem 3, i.e., that for all $\alpha, \beta > 0$ and for any monospline $m^h_{\kappa^i}$ we have

$$E_0(S_h(m_{n,r}))_{1;\alpha,\beta} \leqslant E_0(m^h_{\kappa^i})_{1;\alpha,\beta}. \tag{7.4}$$

However, by the duality Theorem 9 and the representation (1.4) for $R(f^h, \kappa)$, we see that if the monospline $m^h_{\kappa^i}$ corresponds to the quadrature formula $\kappa^i \in K^i_n(h)$, then

$$E_0(m^h_{\kappa^i})_{1;\alpha,\beta} = R^+(W^r_{\infty;\alpha^{-1},\beta^{-1}}; \kappa^i).$$

From this and from Theorem 4 (since $S_h(m_{n,r})$ corresponds to the formula κ^i_n), inequality (7.4) follows, and Theorems 3, 4 are proved.

Now we shall prove Theorem 1. We obtain from relation (1.4) and Theorems 2, 10 and 11 that

$$\begin{aligned} R^{\pm}(W^r F, \kappa^i) &= R^{\pm}(S_h(W^r F), \kappa) = \sup\left\{\int_0^{2\pi} (\pm f(t)) S_h(m)(t)\,dt : f \in F, f \perp 1\right\} \\ &= \sup\left\{\sup_{g:\Pi(g)=\Pi(f)} \int_0^{2\pi} (\pm g(t)) S_h(m)(t)\,dt : f \in F, f \perp 1\right\} \\ &= \sup\left\{\int_0^{2\pi} \Pi(\pm f, t)\Pi(S_h(m), t)\,dt : f \in F, f \perp 1\right\} \\ &\geqslant \sup\left\{\int_0^{2\pi} \Pi(\pm f, t)\Pi(S_h(m_{n,r}), t)\,dt : f \in F, f \perp 1\right\} \\ &= R^{\pm}(S_h(W^r F), \kappa_n) = R^{\pm}(W^r F, \kappa^i_n). \end{aligned}$$

Thus, Theorem 1 is proved.

References

[1] P.S. Aleksandrov, Combinatorial Topology, OGIZ, Moscow, 1947 (in Russian); P.S. Aleksandrov, Combinatorial Topology, vol. 1, Graylock Press, Albany, NY, 1956 (in English).
[2] V.F. Babenko, Nonsymmetric approximations in the spaces of summable functions, Ukrainian Math. J. 34 (1982) 409–416 (in Russian).
[3] V.F. Babenko, Inequalities for rearrangements of differentiable periodic functions, problems of approximation and integrating, Dokl. USSR 272 (1983) 1038–1041 (in Russian).
[4] V.F. Babenko, On a certain problem of optimization of the approximate integration, Studies on Modern Problems of Summation and Approximation of Functions and their Applications, Dnepropetrovsk University, Dnepropetrovsk, 1984, pp. 3–13 (in Russian).
[5] V.F. Babenko, Approximations, widths and optimal quadrature formulae for classes of periodic functions with rearrangement invariant sets of derivatives, Anal. Math. 13 (1987) 15–28.
[6] V.F. Babenko, Widths and optimal quadrature formulae for convolution classes, Ukrainian Math. J. 43 (1991) 1135–1148.
[7] S.V. Borodachov, On optimization of interval quadrature formulae on some nonsymmetric classes of periodic functions, Bull. Dnepropetrovsk Univ. Math. 4 (1999) 19–24 (in Russian).
[8] S.V. Borodachov, On optimization of interval quadrature formulae on some classes of absolutely continuous functions, Bull. Dnepropetrovsk Univ. Math. 5 (2000) 28–34 (in Russian).
[9] N.P. Korneichuk, Extremal Problems of Approximation Theory, Nauka, Moscow, 1976, p. 320 (in Russian).
[10] N.P. Korneichuk, A.A. Ligun, V.G. Doronin, Approximation with Constraints, Naukova dumka, Kiev, 1982 (in Russian).
[11] M.A. Krasnosel'skii, Ya.B. Rutickii, Convex Functions and Orlich Spaces, Fizmatgiz, Moscow, 1958 (in Russian).
[12] S.G. Krein, Yu.I. Petunin, E.M. Semenov, Interpolation of Linear Operators, Nauka, Moscow, 1978 (in Russian).

[13] A.L. Kuz'mina, Interval quadrature formulae with multiple node intervals, Izv. Vuzov Math. 7 (1980) 39–44 (in Russian).
[14] A.A. Ligun, Exact inequalities for spline-functions and best quadrature formulae for some classes of functions, Math. Zametki 19 (1976) 913–926 (in Russian).
[15] G.V. Milovanovic, A.S. Cvetkovic, Gauss–Radau and Gauss–Lobatto interval quadrature rules for Jacobi weight function, Numer. Math. 3 (102) (2006) 523–542.
[16] V.P. Motornyi, On the best quadrature formula of the form $\sum_{k=1}^{n} p_k f(x_k)$ for certain classes of periodic differentiable functions, Izv. Akad. Nauk SSSR. Ser. Mat. 38 (1974) 583–614 (in Russian).
[17] V.P. Motornyi, On the best interval quadrature formula in the class of functions with bounded rth derivative, East J. Approx. 4 (1998) 459–478.
[18] M. Omladich, S. Pahor, S. Suhadolc, On a new type of quadrature formulae, Numer. Math. 25 (1976) 421–426.
[19] K.I. Oskolkov, On optimality of quadrature formula with equidistant nodes on the classes of periodic functions, Dokl. Akad Nauk USSR 249 (1979) 49–52 (in Russian).
[20] Fr. Pittnauer, M. Reimer, Interpolation mit Intervallfunctionalen, Math. Z. 146 (1976) 7–15.
[21] R.N. Sharipov, Best interval quadrature formulae for Lipschitz classes, Constructive Function Theory and Functional Analysis, vol. 4, Kazan University, Kazan, 1983, pp. 124–132 (in Russian).
[22] H. Tribel, Theory of Interpolation, Function Spaces, Differential Operators, Mir, Moscow, 1980 (in Russian).
[23] A.A. Zhensykbaev, The best quadrature formula for some classes of periodic functions, Izv. Akad Nauk USSR, Ser. Math. 41 (1977) 1110–1124 (in Russian).
[24] A.A. Zhensykbaev, Monosplines of minimal norm and the best quadrature formulae, Uspehi Math. Nauk. 36 (1981) 107–159 (in Russian).